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Dirac magnetic monopole and the Aharonov–Bohm solenoid in the Poincaré gauge

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Abstract. We consider the Poincaré (or multipolar) gauge with finite and infinite reference point in connection with singularities of the Aharonov–Bohm solenoid and Dirac magnetic monopole type. Families of paths on which the Poincaré gauge potentials are defined may give rise to 'shadow' surfaces or regions on which the Poincaré gauge potentials are singular. These singularities may be avoided by changing the family of paths to another family (based on the same reference point) and this is equivalent to changing the gauge. We consider the Dirac magnetic monopole using the Poincaré gauge with a family of parallel straight paths from reference points situated at $(0, 0, 0, \pm\infty)$, producing the 'overlapping' potentials for the monopole. A method is given for calculating the Poincaré gauge potentials on the shadow surface arising from a singularity, and this is illustrated by considering the solenoid problem in which the solenoid is given a finite radius (ε), and it is shown that the shadow surface in this case contains singularities of the Dirac delta function type.

1. Introduction

In quantum mechanics and in field theory the gauge-invariant replacement $\hat{E} \to \hat{E} - e\phi$, $\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} - e\boldsymbol{A}$ (or the minimal-coupling substitution $\partial^\mu \to \partial^\mu + (\mathrm{i}e/\hbar c)A^\mu$ where $\hat{p}^\mu = \mathrm{i}\hbar\partial^\mu = (\hat{E}/c, \hat{\boldsymbol{p}})$ and $A^\mu = (\phi, c\boldsymbol{A})$), describes the interaction with the electromagnetic field of a charge field (for example, the charged Dirac or Klein–Gordon fields). The total Lagrangian density in which this change is made to the matter fields will then be invariant under the extended or simultaneous gauge transformations

$$\psi \to \psi' = \exp\left(\frac{\mathrm{i}e}{\hbar c}\Lambda\right)\psi \qquad A^\mu \to A'^\mu = A^\mu - \Lambda^{,\mu}$$

of the first and second kinds wherein Λ is called the gauge function. The Lagrangian density may then be written in the usual form with interaction term. In the free-field case, the variation of the Lagrangian density with respect to the potentials yields $F^{\mu\nu}{}_{,\nu} = 0$, in which $F^{\mu\nu} = A^{\mu,\nu} - A^{\nu,\mu}$, the latter equations reducing these equations to $\Box A^\mu = \chi^{,\mu}$ where $\chi = A^\mu{}_{,\mu}$. In classical theories one may choose the Lorentz gauge condition $\chi = 0$, thus reducing Maxwell's equations to $\Box A^\mu = 0$, and gauge invariance within the Lorentz gauge is governed by the equation $\Box\Lambda = 0$.

Amongst other possible gauge conditions, the radiation (or Coulomb) gauge and the axial gauge are well known examples of non-covariant gauges. A less well known gauge than the Lorentz gauge which is relativistically *covariant* is the Poincaré or

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'multipolar' gauge condition (see, for example, Kobe 1982, 1983, Brittin *et al* 1982, Cornish 1984). By choosing a gauge function of the form

$$\Lambda(x)=\int_R^P A^\mu(z)\,\mathrm{d}z_\mu \tag{1}$$

where R is a reference point and P a field point (x), and by choosing the path from R to P to be everywhere spacelike, Healy and others have shown how this gauge condition may be constructed (calling it the multipolar gauge; Healy 1978, 1979, 1980). In equation (1) the gauge for A^μ is initially assumed to be arbitrary. This produces for A'^μ the expression (in Healy's notation):

$$A'^\mu=\int_R^P F_{\alpha\beta}(z)\frac{\partial z^\alpha}{\partial x_\mu}\,\mathrm{d}z^\beta \tag{2}$$

which describes the Poincaré gauge (PG). (Hereafter we use the symbol $\mathscr{P}$ on the four-potential to denote that it is calculated in the PG.) This generalises to finite paths and a finite reference point an old result of DeWitt (1962) (see also Sachs 1948, Mandelstam 1962, Belinfante 1962, Lévy 1964, Rohrlich and Strocci 1965) again for spacelike curves:

$$\underset{\mathscr{P}}{A^\mu}(x)=\int_{-\infty}^{0} F_{\alpha\beta}(z)\frac{\partial z^\alpha}{\partial x_\mu}\frac{\partial z^\beta}{\partial \xi}\,\mathrm{d}\xi \tag{3}$$

wherein, in the more complete notation of DeWitt, the functions $z^\mu(x,\xi)$ represent four arbitrary single-valued differentiable functions of the spacetime coordinates x^μ and a parameter ξ, which are defined for all x^μ and for all values of ξ in the interval $-\infty<\xi\leqslant 0$, and satisfy the boundary conditions $z^\mu(x,0)=x^\mu$, $\lim_{\xi\to-\infty} z^\mu(x,\xi)=$ spatial infinity. The reference point taken at spatial infinity simplifies the gauge since the potentials and fields are assumed to vanish there. Healy's gauge function (1) is the function

$$\Lambda(x)=\int_0^1 A^\mu(z)\frac{\partial z_\mu}{\partial \xi}\,\mathrm{d}\xi$$

in DeWitt's notation, in which $z^\mu(x,\xi)$ satisfies the conditions $z^\mu(x,1)=x^\mu$, $z^\mu(x,0)=$ coordinates of the reference point R, rather than those of DeWitt, this then being a straightforward generalisation of the infinite-path case to finite paths. The PG representation (2) with finite reference point is then

$$\underset{\mathscr{P}}{A^\mu}(x)=\int_0^1 F_{\alpha\beta}(z)\frac{\partial z^\alpha}{\partial x_\mu}\frac{\partial z^\beta}{\partial \xi}\,\mathrm{d}\xi \tag{4}$$

again in DeWitt's notation.

It is easily shown that the following is an *identity* for $A^\mu(x)$ in any gauge:

$$A^\mu(x)=\partial^\mu\int_0^1 A^\nu(z)\frac{\partial z_\nu}{\partial \xi}\,\mathrm{d}\xi+\int_0^1 F_{\alpha\beta}(z)\frac{\partial z^\alpha}{\partial x_\mu}\frac{\partial z^\beta}{\partial \xi}\,\mathrm{d}\xi. \tag{5}$$

This identity holds for any path (which may be timelike). Each term of the identity depends upon the path, its variation to adjacent points (because of the derivatives of the z) and the endpoints. When the gradient term on the right-hand side is omitted and the integral is taken to *be* the potential, this sets the PG so to speak. Besides being manifestly Lorentz covariant, one other advantage of the PG is that the potentials (and the charge fields in the quantum mechanical description) may be formulated in terms of field strengths only, but this touches on a thorny problem as to whether A^μ is itself to be regarded as a physical field in quantum mechanics (see, for example, Aharonov and Bohm (1959, 1961, 1962, 1963), but more particularly Philippidis *et al* (1982); see also Boyer (1972), Yang (1976), Nieto (1984); for a review paper see Erlichson (1970)). A disadvantage that some might see is in having the fields arise *non-locally* in the line integrals. If the integration path to the field point x is changed to another (spacelike) path to x, Healy and others have shown that this induces a gauge transformation on the potentials (i.e. within the PG). In other work in which the gauge is called the PG it has been found that the formalism is related in the straight-path case essentially to three well known classical vector analysis results (the first due to Liebmann 1908, and the others due to Brand 1950). This increases our perspective of the PG.

It may be shown that by choosing *timelike* curvilinear (world-line) path integrals, rather than spacelike ones and by using a proper-time formalism (which is justified in a relativistic description), the PG reduces in most instances to the Lorentz gauge condition provided one chooses a linear dependence on the endpoint in the functions $z^\mu(x, \xi)$ (which are in any case too wide):

$$z^\mu(x, \xi) = x^\mu + z^\mu(\xi). \tag{6}$$

The sole criterion for distinguishing whether the Lorentz gauge condition is, in fact, satisfied is then governed by insisting that the integration paths do not cross current distributions for $F^{\mu\nu}$. We shall continue this work later.

Alongside this we note the work which has been carried out on the Aharonov-Bohm (AB) effect which this formalism may be used to interpret. We refer, for example, to the articles by Roy (1980) and by Zuchelli (1984). The exemplary paper by Wu and Yang (1975) is the generally accepted explanation of the AB effect, but this paper did not touch on the application of the PG (or multipolar gauge) to this problem. The paper by Roy, which paradoxically set out to *disprove* the non-locality of the fields arising in the AB effect on the basis, effectively, of the use of the equations in the PG, has been generally criticised. The criticisms made by Zuchelli of Roy's paper, however, were not constructive, leaving some doubt, even, that the use of the equations in the PG may be relevant. One of the criticisms of Roy's work by Zuchelli was that the paths $z^\mu(x, \xi)$ had to come from spatial infinity and so would have to cross the return path of the flux (omitted in Roy's treatment) and this flux could be arranged to be inside a sphere, the whole experiment being performed in its interior. However, given that the criticism applies strictly to Roy's statements, the work may be generalised to cover the case where the paths $z^\mu(x, \xi)$ cross current distributions for $F^{\mu\nu}$ and these may be so arranged to have a finite starting reference point rather than an infinite one. The second criticism that Zuchelli made was more substantial, but again had to do with the strict statement made by Roy. Taking the situation in its wider application it is unusual that neither Zuchelli nor Roy spotted that a change of path (along which the potential vanished) actually implied a change of gauge at the instant the current distributions are crossed, so that a valid explanation of the AB effect is possible.

As an example of the PG, we shall apply the gauge (in section 2) to the AB solenoid and (in section 3), more substantially, to the construction of the potentials for a Dirac magnetic monopole situated at the origin using parallel transport with an infinite reference point referred to above (i.e. equation (6)), which would be expected to produce a Lorentz gauge potential, but we do this for spacelike paths rather than timelike ones. Specifically, we choose two infinite reference points at $z = \pm\infty$ and reproduce the 'overlapping' potentials for the monopole given by Wu and Yang in their explanation of how spurious singularities (i.e. not arising from the position of the monopole itself) may be eliminated by simultaneous gauge transformations. We explain how the singularities arise as 'shadows' of the monopole in the treatment using these families of paths and in section 4 we indicate a method for finding the PG potentials in the shadow region.

2. The Aharonov–Bohm solenoid in the Poincaré gauge with finite or infinite reference point

Potentials for the solenoid may be approximated using cylindrical polar coordinates either by

$$\begin{gathered}\phi=0 \qquad A_\rho=0 \qquad A_z=0 \\ A_\phi=\frac{\Phi}{4\pi\rho}\left(\frac{L-z}{[(L-z)^2+\rho^2]^{1/2}}+\frac{L+z}{[(L+z)^2+\rho^2]^{1/2}}\right)\end{gathered} \tag{7}$$

or by

$$\phi=0 \qquad A_\rho=0 \qquad A_\phi=\frac{\Phi}{2\pi\rho} \qquad A_z=0. \tag{8}$$

The potentials in (7) are the potentials for a solenoid of zero radius extending from $z=-L$ to $z=L$ (from the addition of infinitesimal circular current elements; Roy 1980) and are the same as the potentials for a 'line' magnet of length $2L$ lying along the z axis. Potentials (8) are the limiting potentials arising from (7) in the limit $L\to\infty$. In either case there is a line singularity in the vector potential (along the z axis) around which at infinitesimal distances in the xOy plane $\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}=\Phi$, but whereas in (8) there is no return flux, in (7) there is one, revealed by $\oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{r}$, taken round a circle of radius ρ in the xOy plane, reducing in value from Φ to 0, as ρ increases from 0 to ∞. Thus we meet these singularities in the form of line singularities, the difficulties associated with which we shall try to remove from the PG.

It is straightforward that (8), where the fields vanish outside the solenoid, satisfies identity (5) (extended to include the case of an infinite reference point if necessary) with the final term vanishing, provided the domain covered by the endpoint x does not include the endpoints reached by paths passing through the singularity. Thus the PG potential (the final term of (5)) based on the specific family of paths $z^\mu(x,\xi)$ vanishes everywhere except in the 'shadow' region of the solenoid—i.e. in the region extending from the solenoid containing endpoints of paths $z^\mu(x,\xi)$ which have crossed the solenoid (this region is the time evolution of a semi-infinite surface). Since, in the domain considered, every point x is the endpoint of a unique member of the family and there is no other member of the family having this endpoint, we may imagine the family (or rather a family which we would normally consider) to be rather like a curved beam of light originating from a point source (R)†, so that the shadow region of the solenoid will indeed be the (time evolution of the) two-dimensional surface indicated.

† This may not always be the case but the general conclusions remain: the elimination from $z^\mu(x,\xi)=s^\mu+\eta t^\mu+\zeta u^\mu$ with s, t, u fixed, of the variables ξ, η, ζ leaves one equation connecting the x, i.e. the shadow region will be a three-dimensional hypersurface. ($s^\mu+\eta t^\mu+\zeta u^\mu$ represents the time evolution of the solenoid.)

For all points x on this surface the PG potential will diverge. However, provided a path $z^{\mu}(x, \xi)$ is clear of the singularity (with x not in the shadow region)

$$\Lambda(x) = \int_{R}^{P(\phi)} A^{\mu}(z)\,\mathrm{d}z_{\mu} = -\int_{0}^{\phi} cA_{\phi}\rho\,\mathrm{d}\phi = -c\Phi\phi/2\pi$$

with

$$\Lambda_{,\mu} = (0, 0, c\Phi/2\pi\rho, 0) = A_{\mu}.$$

Note that we are not allowed to write zero for the PG potentials on the shadow surface (SS_1) itself arising from *this* family of paths (P_1), but we may use another family of paths (P_2) starting from the same reference point (using a different PG potential with a different shadow region) to obtain zero PG potentials for the new family of paths (see figure 1). We use a gauge transformation in going from the paths P_1 to the paths P_2, but for equations (8), the fact that at x there is no difference between the PG potentials based upon the paths P_1 and upon the paths P_2 is because the flux contained within the paths is a constant independent of x. Expressing this differently, for any x there will be two paths arriving there from R and as x is varied taking the paths with it the flux contained within the paths does not change. All of this derives from the subtraction of (5) for P_2, from (5) for P_1, noting that A^{μ} is single-valued over $P_1\bar{P}_2$. Because a solenoid would have non-zero diameter, some return flux and non-constant current, this idealisation in which we assume that P_1 and P_2 conform to the two halves of the electron beam (as in any idealisation of the AB effect) could not be applied in practice; it may provide only the basis of a first approximation. The main point is that it takes a gauge transformation to go from the PG based upon P_1 to the PG based upon P_2. Thus the fringe-shift phenomenon of the AB effect could be derived using the PG from the effects of the simultaneous gauge transformation (in which the electromagnetic transformation is the identity $\underset{\mathscr{P}}{A_1}{}^{\mu} = \underset{\mathscr{P}}{A_2}{}^{\mu}$ in the PG in the first approximation). This gauge transformation taking one set of paths into another set (in this simplified picture concerning equation (8)) may be considered as the specialisation of the transformation which uses two reference points R_1 and R_2 with associated paths P_1 and P_2 (figure 2). The gauge function for this transformation is the sum of the flux calculated over R_2R_1x and the line integral $\int A^{\mu}\,\mathrm{d}x_{\mu}$ taken along the arbitrary curve R_1R_2. In addition to using these possibilities with the PG it is also possible to use moving reference points (Healy 1980).

Using the PG to compare the potentials via paths on either side of the solenoid in this way, using different families, should be done, more precisely, for the paths P_1, P_2 in spacetime, where R is in the past relative to the endpoint; but since the situation

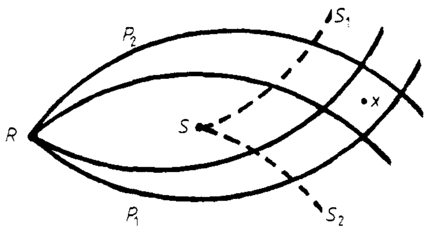

Figure 1. Changing the family of paths P_1 to the family of paths P_2 using the same reference point. S is the singularity (giving rise to shadow surfaces SS_1 and SS_2).

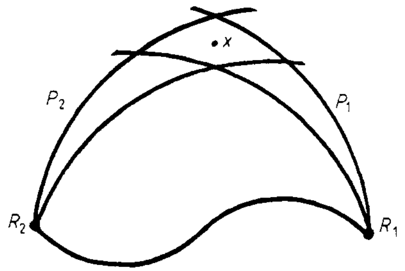

Figure 2. Changing the family of paths P_1 to the family of paths P_2 using a different reference point.

here is static we have omitted this. Instead of choosing different families on either side of the singularity to explore the potential, an alternative method is to use the same family of paths (for example a family of parallel straight lines from an infinite reference point). We are not then comparing potentials at the same point x but are using the same family to compare potentials at adjacent points on either side of the shadow region. This method illustrates the PG when there are line singularities. Within the same family of paths (figure 3) we have P_1 and P_2 on either side of the singularity going to adjacent points x, x'. If $x'=x+\mathrm{d}X$ where $\mathrm{d}X$ is arbitrary but fixed as x varies, and the infinitesimal curvilinear triangular-shaped surface bounded by P_1 and P_2 intersects the singularity at S, we may take the portion (from R to S) of the unique path of the family which goes from R to the shadow surface on xx' and also use this as the return path from S, after surrounding S with an infinitesimal closed curve γ. Then from (5), calling the complete (closed anticlockwise) circuit enclosing the non-singular region for the potentials $\partial\Sigma$, we have

$$\underset{\mathscr{P}}{A^\mu}(x)-\underset{\mathscr{P}}{A^\mu}(x')=-\partial^\mu\left(\oint_{\partial\Sigma} A^\nu(z)\,\mathrm{d}z_\nu+\int_\gamma A^\nu(z)\,\mathrm{d}z_\nu-A^\nu(x)\,\mathrm{d}X_\nu\right)+A^\mu(x)-A^\mu(x').$$

This equation reduces to

$$\underset{\mathscr{P}}{A^\mu}(x)-\underset{\mathscr{P}}{A^\mu}(x')=-F^{\mu\nu}(x)\,\mathrm{d}X_\nu-\partial^\mu\oint_\gamma A^\nu(z)\,\mathrm{d}z_\nu-\partial^\mu\iint_\Sigma \tfrac{1}{2}F^{\alpha\beta}(\omega)\,\mathrm{d}\sigma_{\alpha\beta}$$

where the line integral represents the singular flux (at S) and the double integral represents the return flux. This equation compares the potentials at x and x' in the PG based on the same family of paths. Applying this equation to the simple case of

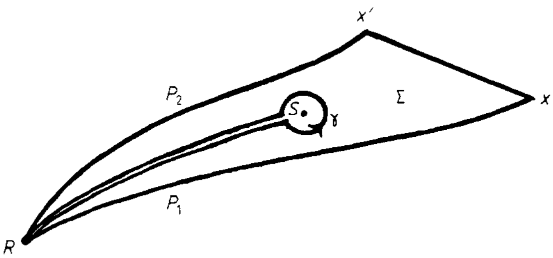

Figure 3. Comparing the potentials (using the same family of paths) at the events x, x' on either side of the shadow surface originating from S.

a set of parallel paths $z^\mu(x, \xi) = (ct, x+\xi, y, z)$, where ξ varies from $-\infty$ to 0, and the solenoid is as described by equations (8), we may take the interval $x'-x$ as finite in this instance; then, since there is no return flux

$$\begin{aligned} \underset{\mathscr{P}}{A}^\mu(x) - \underset{\mathscr{P}}{A}^\mu(x') &= \int_{P_1} F^{\mu\nu}(z)\,\mathrm{d}z_\nu - \int_{P_2} F^{\mu\nu}(z)\,\mathrm{d}z_\nu \\ &= -\partial^\mu\{\Lambda\theta(-y)\} \\ &= (0, 0, -\Lambda\delta(y), 0) \end{aligned}$$

in the vicinity of the shadow surface, where $\Phi = -\Lambda/c$ is the constant singular flux and $\theta(y)$, $\delta(y)$ are the unit step and Dirac delta functions in y. Since the PG potentials are zero everywhere except on the shadow surface $y = 0$, where they have the value $(0, 0, -\Lambda\delta(y), 0)$, all of these values allow us to have $\oint \underset{\mathscr{P}}{A}^\mu(z)\,\mathrm{d}z_\mu$, taken round the semi-infinite trapezoidal circuit, equal to Λ when the singularity is enclosed, or zero when it is not. Assuming that the example may be generalised, we conclude that the PG may be used in the same way as other gauge potentials in the minimal-coupling formula if required.

3. The Dirac magnetic monopole in the Poincaré gauge with infinite reference point

The 'overlapping' Lorentz gauge potentials for the monopole of strength g situated at the origin, $\boldsymbol{r} = \boldsymbol{0}$, for which $\boldsymbol{B}(\boldsymbol{r}) = g\boldsymbol{r}/r^3$, are, using spherical polar coordinates:

$$\phi = 0 \qquad A_r = A_\theta = 0 \qquad A_\phi = \frac{g}{r\sin\theta}(1-\cos\theta) \tag{9}$$

$$\phi = 0 \qquad A_r = A_\theta = 0 \qquad A_\phi = \frac{-g}{r\sin\theta}(1+\cos\theta). \tag{10}$$

The potentials (9) are singular along the negative z axis, and (10) are singular along the positive z axis. However, a (simultaneous) gauge transformation may be employed taking (9) into (10) (for the respective regions which exclude the singularities, i.e. for the overlapping region $\frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi + \delta$, where $0 < \delta \leqslant \frac{1}{2}\pi$), and the whole of space-time minus the position of the monopole may be covered by one or other of the potentials (9), (10), either potential being suitable in the singular-free region associated with it to describe the fields because their curls are equal to the magnetic field there. Assuming that the phase factor in the transformation of the wavefunction, $\exp[(2ige/\hbar)\phi]$, is single valued leads to the Dirac quantisation condition for the monopole (Wu and Yang 1975), but we are interested in the electromagnetic component here.

Consider the monopole to be 'illuminated' by parallel straight paths having a reference point, R, at $z = -\infty$, i.e. consider paths that are given by

$$z^\mu(x, \xi) = x^\mu + \xi k^\mu \qquad -\infty < \xi \leqslant 0$$

where $k^\mu = (0, 0, 0, 1)$. Since $\partial z^\alpha/\partial x^\mu = \delta^\alpha_\mu$, we have for the PG potential,

$$\underset{\mathscr{P}}{A}_\mu(x) = \int_R^P F_{\alpha\beta}(z)\frac{\partial z^\alpha}{\partial x^\mu}\,\mathrm{d}z^\beta = \int_{-\infty}^0 F_{\mu\beta}(z)\frac{\mathrm{d}z^\beta}{\mathrm{d}\xi}\,\mathrm{d}\xi = \int_{-\infty}^0 F_{\mu 3}(z)\,\mathrm{d}\xi$$

provided we exclude from the domain covered by the event x the shadow region of the monopole (i.e. the hypersurface $x=y=0$, $z\geqslant 0$, all t). Thus

$$\underset{\mathscr{P}}{\phi}(x)=0 \qquad c\underset{\mathscr{P}}{\boldsymbol{A}}(x)=\int_{-\infty}^{0}(F^{31}(z),-F^{23}(z),0)\,\mathrm{d}\xi.$$

Using the complex variables

$$\underset{\mathscr{P}}{\mathscr{A}}(x)=\underset{\mathscr{P}}{A_x}(x)+\mathrm{i}\,\underset{\mathscr{P}}{A_y}(x) \qquad \mathscr{B}(z)=B_x(z)+\mathrm{i}B_y(z) \qquad \mathscr{X}=x+\mathrm{i}y$$

where x and y in the last equation are coordinates, we have

$$\begin{aligned}\underset{\mathscr{P}}{\mathscr{A}}(x)&=-\mathrm{i}\int_{-\infty}^{0}\mathscr{B}(z(x,\xi))\,\mathrm{d}\xi\\ &=-\mathrm{i}g\mathscr{X}\int_{-\infty}^{0}\frac{\mathrm{d}\xi}{[|\mathscr{X}|^2+(z+\xi)^2]^{3/2}}\\ &=-\mathrm{i}g\frac{\mathscr{X}}{|\mathscr{X}|^2}\left[\frac{z+\xi}{[|\mathscr{X}|^2+(z+\xi)^2]^{1/2}}\right]_{-\infty}^{0}\\ &=-\mathrm{i}g\frac{\mathscr{X}}{|\mathscr{X}|^2}\left(\frac{z}{(|\mathscr{X}|^2+z^2)^{1/2}}+1\right)\\ &=-\frac{\mathrm{i}g\exp(\mathrm{i}\phi)}{r}\,\frac{(1+\cos\theta)}{\sin\theta}.\end{aligned}$$

Hence we have shown that the potentials $\underset{\mathscr{P}}{A^{\mu}}(x)$ in this case are the potentials (10), and it is not accidental that the line of singularities along the positive z axis corresponds to the shadow region of the monopole. Equally, it may be shown, by taking R at $z=\infty$, that the other potentials (9) result, in that case the shadow region is along the negative z axis.

4. Integrating 'through' a singularity to find the PG potentials in the shadow region

In the calculation for the PG potentials of the monopole we excluded from the domain covered by x the shadow region of the monopole. If we follow through the above calculation with x in the shadow region we obtain zero for the PG potentials in the exact direction of the z axis (excluding the monopole's position) but this depends on a subtraction of infinities where the path crosses the monopole. A more correct method for obtaining the PG potentials in the shadow region would be to calculate the potentials in the shadow region of a monopole of finite radius ε in which paths are allowed to pass through the monopole (in this, two four-potentials would be required), and then to let ε decrease to zero. The calculation for the PG potential in the shadow region of the solenoid (see section 2) by this method is simpler and more interesting. We shall use it to illustrate the method.

Taking a solenoid of finite radius ε with its axis lying along the z axis for which $B_z=\Phi/\pi\varepsilon^2$ (Φ constant) within the solenoid ($B_z=0$ outside), we have in the notation for the monopole problem after changing axis notation:

$$\underset{\mathscr{P}}{\mathscr{A}}(x)=-\mathrm{i}\int_{-\infty}^{0}\mathscr{B}(z(x,\xi))\,\mathrm{d}\xi \qquad \text{where} \qquad \mathscr{B}(z)=B_y(z)+\mathrm{i}B_z(z)$$

i.e. $\underset{\mathscr{P}}{A}_x(x)=\underset{\mathscr{P}}{A}_z(x)=0$ and

$$\underset{\mathscr{P}}{A}_y(x)=\int_{-\infty}^{0} B_z(z(x,\xi))\,\mathrm{d}\xi=\frac{2\Phi}{\pi\varepsilon^2}\,\theta(x)(\varepsilon^2-y^2)^{1/2}\theta(\varepsilon^2-y^2)$$

when the event x is exterior to the solenoid (note that the x and y on the right-hand side of the last equation are coordinates). For a solenoid of zero radius the PG potentials everywhere (excluding the position of the solenoid itself) are thus given by

$$\underset{\mathscr{P}}{\phi}=0 \qquad \underset{\mathscr{P}}{\boldsymbol{A}}=\left(0,\,\Phi\theta(x)\lim_{\varepsilon\to 0}\left\{\frac{2}{\pi\varepsilon^2}(\varepsilon^2-y^2)^{1/2}\theta(\varepsilon^2-y^2)\right\},0\right)$$

and we may confirm that the above limit is an alternative representation of the Dirac delta function $\delta(y)$ as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}\delta(y)\,\mathrm{d}y&=\lim_{\varepsilon\to 0}\left(\frac{2}{\pi\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}(\varepsilon^2-y^2)^{1/2}\,\mathrm{d}y\right)\\&=\frac{2}{\pi}\lim_{\varepsilon\to 0}\left[\frac{1}{2}\frac{y}{\varepsilon}\left(1-\frac{y^2}{\varepsilon^2}\right)^{1/2}+\tfrac{1}{2}\sin^{-1}\left(\frac{y}{\varepsilon}\right)\right]_{-\varepsilon}^{\varepsilon}\\&=1.\end{aligned}$$

The PG potentials for the solenoid in the limiting situation are thus

$$\underset{\mathscr{P}}{\phi}=0 \qquad \underset{\mathscr{P}}{\boldsymbol{A}}=(0,\Phi\theta(x)\delta(y),0)$$

as previously calculated in section 2.

5. Conclusion

The difficulties associated with line- and point-type singularities may be removed from the PG, contrary to previous suggestions. We have shown that the PG may be used to describe the AB solenoid, and we have also applied the PG to produce the well known 'overlapping' potentials of the monopole. The methods we have given for calculating the PG potentials in the shadow region should be applicable to the monopole, and it should be possible to relate the gauge transformation existing between the 'overlapping' potentials to that of the scheme described by figure 2 in which the families originate from two different reference points R_1, R_2 (these would be taken at $z=\pm\infty$). It would be necessary to have an explicit calculation for the two shadow regions before attempting this calculation. We believe that the PG may be used in the minimal-coupling substitution in the presence of singularities, and we are surprised to have found that the gauge does not appear to be particularly well known.

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